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Hypergroups over the group

SYNOPSIS

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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանի մաթեմատիկայի և մեխանիկայի ֆակուլտետի խորհրդի կողմից:

Գիտական ղեկավար՝ ֆիզ.-մաթ. գիտ. դոկտոր
Ս. Հ. Դալալյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ.-մաթ. գիտ. դոկտոր
Վ. Ս. Աթաբեկյան
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Վ. Հ. Միքայելյան
ֆիզ.-մաթ. գիտ. թեկնածու
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Առաջատար կազմակերպություն՝ Հայ-Ռուսական համալսարան

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The topic of the thesis was approved in Yerevan State University

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The defense will be held on July 4, 2023 at 15 : 00 at a meeting of the specialized council of mathematics 050, operating at the Yerevan State University (0025, 1 Alek Manukyan St, Yerevan).
The thesis can be found at the YSU library.
The synopsis was sent on May 25, 2023.

Scientific secretary of specialized council K. L. Avetisyan

General characteristics of the work

Actuality of the theme. The concept of hypergroup over the group naturally arises when one tries to extend the concept of quotient group for the case of an arbitrary subgroup. Instead of the naturally induced binary multiplication operation on the right quotient-set $N\backslash G$ (for a group G and its normal subgroup N) in the case of arbitrary subgroup H one can define an analogous operation on a right transversal M of the subgroup H in the group G, i.e. on a subset $M \subseteq G$ with $|M \cap Hg| = 1$ for all $g \in G$. Such an operation on M is considered in several previous research [3, 20, 31][1] and maps the element pair $(a, b) \in M \times M$ to such an element $c \in M$ that $c \in H(ab)$.

It can be proven that the set M with the binary operation defined above is a right quasigroup with a left neutral element and becomes a group isomorphic to the quotient-group G/H if H is a normal subgroup. In the section 3.4 of the dissertation we show (see Theorem 3.7, Corollary 3.7.1 in the dissertation) that any right quasigroup possessing a left neutral element can be obtained up to isomorphism from a triple (G, H, M) by the way mentioned above (see also [3] Theorem 1.1, [20] Theorem 3.4). This, in fact, leads to a **Cayley-type** theorem for right quasigroups with a left neutral element describing the latters in terms of permutation group triples (G, H, M) (see Corollary 3.7.1 in the dissertation). The connection between the group triples (G, H, M) and the hypergroups over the group is straightforward after we introduce the standard construction of hypergroups over the group in the chapter 1 of the dissertation. Moreover, the categorical equivalence between the theories of hypergroups over the group and the group triples (G, H, M) can be shown (see [10]). Also, hypergroups over the group, extracting the algebraic structure of triplets (G, H, M), can be investigated from the perspective of universal algebra [5, 8, 33, 13, 25]. An important aspect of investigation in universal algebra is the exploration of hyperidentites [27, 28], which can lead to new directions in the theory of hypergroups over the group.

Originally formulated for groups, Cayley-type theorems have later been extended for such concepts as semigroups [7], dimonoids [37, 36], or g-dimonoids [35]. As we mentioned above, this list can be expanded with a Cayley-type theorem for right quasigroups with a left neutral element, and, as a consequence, for loops. Such a theorem allows to investigate the right quasigroups with a left neutral element by means of group theory, particularly the theory of group triples (G, H, M). Indeed, in [31] Niemenmaa and Kepka used the concept of (G, H, M) triples (in their case $M = G/H$) to prove a necessary and sufficient condition (Theorem 4.1, see also Theorem 5.1 in [32]) for a group G to be the multiplication group

[1] In [31], preserving the concept, an analogous operation was introduced on the right quotient-set G/H instead of M.

(introduced by Albert in [1] and [2]) of a loop. As a corollary (Theorem 5.2 in [32]) it can be shown that any non-trivial finite cyclic group can not be the inner mapping group (introduced by Bruck in [6]) of a loop. In [3] Baer considered a special class of (G, H, M) triples to describe all nets up to isomorphism in terms of the isotopy (called similarity in the original paper) classes of loops (Theorems 6.2 (a), 6.3 and 7.1). For more details on the theory of quasigroups and loops, and isotopy relation see [4].

In [14] Kuznetsov used (G, H, M) triples to get a necessary and sufficient condition (Theorem 1) for the existence of sharply k-transitive loops of permutations on a finite set X with the cardinality $n \geq k$. Using the special case when $k = 2$ Kuznetsov described (see [19], Remark 1) all projective planes of order n (where n is a prime power) by means of derived loop transversals in $G = S_n$ of its subgroup $H = St_{a,b}(S_n)$ (the subgroup of all permutations fixing the elements $a, b \in \{1, 2, \dots, n\}$), where a, b are arbitrary. Moreover, in his previous work [15, 16, 17, 18, 14, 19] Kuznetsov developed the theory of transversals in groups, mainly considering the special case when the largest normal subgroup of G contained in the subgroup H is trivial, i.e. $Core_G(H) = \{1\}$. This special case introduces several interesting properties for transversals which are also reflected in the theory of hypergroups over the group in the dissertation.

As we have mentioned above, previous research on using the group triples (G, H, M) (or, equivalently, hypergroups over the group) was more focused on the right quasigroup structure Ξ on right transversals M, whereas the concept of hypergroups over the group encapsulates also more connections between the group H and the right quasigroup (M, Ξ). These connections are reflected in the structural mappings and the eight axioms of hypergroups over the group and are proven to be enough (see Theorem 1.3 in the dissertation) for constructing the initial group G from its subgroup H and the given transversal M. Thus, the hypergroups over the group allow to construct new groups G based on a group H and a right quasigroup (with a left neutral element) (M, Ξ). For this one needs to define the structural mappings of a corresponding hypergroup over the group that satisfy the eight axioms. In the dissertation it is shown that the mentioned axioms are independent, however for some special cases four of them follow from the other four by so making the construction of the group G much more easier.

Let us emphasize that R. Lal has introduced the concept of c-groupoids [20] which coincides with a special case of the hypergroups over the group, namely the unitary hypergroups over the group [11]. In the chapter 2 of the dissertation we describe all hypergroups over the group up to isomorphism in terms of unitary hypergroups over the group. The concept of c-groupoids has found its applications in a characterization of Tarski monsters [22], in a research on right transversals in topological groups [23], and in other problems. Particularly, by using c-groupoids, in [21] Lal and Shukla have shown the equivalence between the concepts of normal and perfectly stable subgroups of a group in the case of finite groups. However the question on this equivalence in general (infinite) case still remains open (see [20]). In the dissertation we define a similar concept of completely stable subgroup and prove a similar Theorem 4.1 for both finite and infinite cases. Our proof is conducted by using the concept of hypergroups over the group which has arisen independently from the concept of c-groupoids.

Also let us mention that the term "hypergroup" is already used for another concept (see [26, 34], also [24]). Here this concept of hypergroup is not considered, so sometimes we will call the hypergroups over the group shortly hypergroups.

The aim of the dissertation: The main goal of the dissertation is to develop the theory of hypergroups over the group, show some applications and emphasize several directions for further development. Specifically we aimed to

1. describe all hypergroups over the group up to isomorphism by using a special class, namely unitary hypergroups over the group;
2. show the independence of the axiomatic system of hypergroups over the group;
3. investigate the connections between the axioms of hypergroups over the group;
4. describe a method to obtain new groups from a given group and a right quasigroup with a left neutral element (we call it a group extension by a right quasigroup with a left neutral element);
5. describe right quasigroups with a left neutral element in terms of permutation groups (by using hypergroups over the group), in essence by proving a **Cayley-type theorem** for right quasigroups with a left neutral element;
6. investigate connections between the concepts of **normal** and **completely stable** subgroups of a group by using the concept of hypergroups over the group.

The methods of investigations: In the dissertation results of group and quasigroup theories are mainly used.

Scientific Innovation: All the main results are novel.

Theoretical and practical value: The results of the work have theoretical character. The results of the dissertation can be used in the group theory, as well as in the quasigroup theory.

The approbation of the results All results of the thesis are published in scientific journals and/or presented in the following scientific conferences:

1. Navasardyan, Shant. "On complete reducibility of hypergroups over the group", Yerevan State University, Annual Scientific Session of Student Scientific Society, pp. 195-203, April 25-29, 2016, Yerevan, Armenia.

2. Navasardyan, Shant. “On the Axioms of Hypergroups over a Group”, Harmonic Analysis and Approximations, VII (Dedicated to 90th Anniversary of Alexandr Talalyan), pp. 85-87, September 16-22, 2018, Tsaghkadzor, Armenia.
3. Navasardyan, Shant. “The Independence of Axioms of Hypergroup over Group”, Yerevan State University, Emil Artin International Conference (Dedicated to the 120th Anniversary of Emil Artin), pp. 99-100, May 27 - June 2, 2018, Yerevan, Armenia.
4. Navasardyan, Shant. “The independence of Axioms of Hypergroup over Group”, 56th Summer School on Algebra and Ordered Sets, p. 20, September 2-7, 2018, Spindleruv Mlyn, Czech Republic.

Publications: The main results of this dissertation are published in four conference abstracts mentioned above, and four scientific articles which are mentioned at the end of this synopsis.

The structure and volume of the dissertation: The thesis is consisted of introduction, four chapters, conclusion and a list of references. The publications of the author is four conference abstracts, and four articles. The number of references is 47. The volume of the thesis is 77 pages.

The main content of the dissertation

Chapter 1: definition and preliminary results on hypergroups over the group. Let G be a group, H be its subgroup, and $M \subseteq G$ be a right transversal of H in G. Then the following elements can be uniquely represented as multiplications of elements from H and M:

$(St1)$ $a \cdot \alpha = {}^{a}\alpha \cdot a^{\alpha}$,

$(St2)$ $a \cdot b = (a, b) \cdot [a, b]$,

where ${}^{a}\alpha, (a, b) \in H$ and $a^{\alpha}, [a, b] \in M$. Hence the following mappings can be defined:

(Φ) $\Phi : M \times H \to M$, $\quad \Phi(a, \alpha) = a^{\alpha}$, $\quad (\Psi)$ $\Psi : M \times H \to H$, $\quad \Psi(a, \alpha) = {}^{a}\alpha$,

(Ξ) $\Xi : M \times M \to M$, $\quad \Xi(a, b) = [a, b]$ $\quad (\Lambda)$ $\Lambda : M \times M \to H$, $\quad \Lambda(a, b) = (a, b)$.

It can be proved ([9], Theorem 2, [38], Theorem 1, [12], Theorem 2.2) that the mappings Φ, Ψ, Ξ, Λ satisfy the following conditions.

$(P1)$ The mapping Ξ is a binary operation on M such that (M, Ξ) is a **right quasigroup with a left neutral element**, i.e.

- (i) any equation $[x, a] = b$ with elements $a, b \in M$ has a unique solution in M;
- (ii) (M, Ξ) has a left neutral element o, i.e. $[o, a] = a$ for any $a \in M$.

$(P2)$ The mapping Φ is an **action of the group H on the set M**, i.e.

- (i) $(a^{\alpha})^{\beta} = a^{\alpha \cdot \beta}$ for any elements $\alpha, \beta \in H$ and for every $a \in M$;
- (ii) $a^{\varepsilon} = a$ for each $a \in M$, where ε is the neutral element of H.

$(P3)$ For any $\alpha \in H$ there is an element $\beta \in H$ such that $\alpha = {}^{o}\beta$.

$(P4)$ The following identities $(A1) - (A5)$ hold:

$(A1)$ ${}^{a}(\alpha \cdot \beta) = {}^{a}\alpha \cdot {}^{a^{\alpha}}\beta$,

$(A2)$ $[a, b]^{\alpha} = [a^{{}^{b}\alpha}, b^{\alpha}]$,

$(A3)$ $(a, b) \cdot {}^{[a,b]}\alpha = {}^{a}({}^{b}\alpha) \cdot (a^{{}^{b}\alpha}, b^{\alpha})$,

$(A4)$ $[[a, b], c] = [a^{(b,c)}, [b, c]]$,

$(A5)$ $(a, b) \cdot ([a, b], c) = {}^{a}(b, c) \cdot (a^{(b,c)}, [b, c])$.

Definition 1.1 For an arbitrary set M, a group H and a system of mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ we call the triple (M, H, Ω) a (right) **hypergroup over the group** if the conditions $(P1) - (P4)$ are satisfied. Such a hypergroup is denoted by M_H.

Speaking on a hypergroup M_H, we by default denote (if not mentioned otherwise) its system of structural mappings by $\Omega = (\Phi, \Psi, \Xi, \Lambda)$, the neutral element of the group H by ε, the left neutral element of the binary operation Ξ by o, and $\theta = (\Lambda(o, o))^{-1}$.

Definition 1.2 In the case when the hypergroup M_H is obtained by considering a group G, its subgroup H, a transversal M, and the operations Φ, Ψ, Ξ, Λ defined by $(\Phi) - (\Lambda)$ we say that M_H is **obtained by the standard construction from the triple** (G, H, M). Sometimes we call such triples (G, H, M) **group triples**.

Definition 1.3 Let M_H and $M'_{H'}$ be two hypergroups, with the systems of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ and $\Omega' = (\Phi', \Psi', \Xi', \Lambda')$ respectively. A **morphism**

$$f : M_H \to M'_{H'} \tag{1}$$

of hypergroups over the group is a pair $f = (f_0, f_1)$, consisting of a group homomorphism $f_0 : H \to H'$ and a map of sets $f_1 : M \to M'$, such that:

$(M\Phi)$ $\Phi \circ f_1 = (f_1 \times f_0) \circ \Phi'$, $(M\Psi)$ $\Psi \circ f_0 = (f_1 \times f_0) \circ \Psi'$,

$(M\Xi)$ $\Xi \circ f_1 = (f_1 \times f_1) \circ \Xi'$, $(M\Lambda)$ $\Lambda \circ f_0 = (f_1 \times f_1) \circ \Lambda'$.

Naturally, we will call a morphism $f = (f_0, f_1) : M_H \to M'_{H'}$ **isomorphism** if the mappings f_0, f_1 are invertible.

Let H be an arbitrary group, M be an arbitrary set. The set of all hypergroups M_H with the basic set M, defined over the group H, is denoted by $Hg(M, H)$. For the basic set M and the group H, the set of classes of isomorphic hypergroups M_H is denoted by $\mathcal{H}g(M, H)$. An important problem in the theory of hypergroups over the group is to describe the set $\mathcal{H}g(M, H)$.

To show the universality property of the standard construction for a given hypergroup M_H we will construct a group triple $(\overline{G}, \overline{H}, \overline{M})$ such that $\overline{M}_{\overline{H}}$ is isomorphic to M_H.

Consider the Cartesian product $\overline{G} = H \times M = \{\alpha a \mid \alpha \in H, a \in M\}$ and the following binary operation on it:

$$\alpha a \cdot \beta b = (\alpha \cdot {}^{a}\beta \cdot (a^{\beta}, b))[a^{\beta}, b]. \tag{2}$$

It can be shown ([9], Theorem 4) that the set $\overline{G}$ with a binary operation defined by Eq. 2 is a group with the neutral element θo.

Now when we have defined the group $\overline{G}$, let us consuder the following subsets of it:

$$\overline{H} = \{(\alpha \cdot \theta)o \mid \alpha \in H\}, \qquad \overline{M} = \{\varepsilon a \mid a \in M\}. \tag{3}$$

Then (see [9], Theorem 5) $\overline{H} \subseteq \overline{G}$ is a subgroup of the group $\overline{G}$, and $\overline{M}$ is a (right) transversal of $\overline{H}$ in $\overline{G}$. Moreover, the mapping pair

$$f_0 : H \to \overline{H}, \quad \alpha \mapsto (\alpha \cdot \theta)o, \qquad f_1 : M \to \overline{M}, \quad a \mapsto \varepsilon a$$

is an isomorphism between the hypergroups M_H and $\overline{M}_{\overline{H}}$, where $\overline{M}_{\overline{H}}$ is the hypergroup obtained by the standard construction from the group triple $(\overline{G}, \overline{H}, \overline{M})$.

Definition 1.4 The group $\overline{G}$ is called the **exact product** of H and M associated with the hypergroup M_H. We also call $\overline{G}$ the **extension of the group** H **by the right quasigroup (with a left neutral element)** (M, Ξ) associated with the hypergroup M_H.

Definition 1.5 For the hypergroup M_H the structural mappings Φ, Ψ, Ξ, Λ are called **trivial** if respectively

$(Tr\Phi)$ $\Phi(a, \alpha) = a$ for all $a \in M, \alpha \in H$,

$(Tr\Psi)$ $\Psi(a, \alpha) = \alpha$ for all $a \in M, \alpha \in H$,

$(Tr\Xi)$ $\Xi(a, b) = o$ for all $a, b \in M$ and, consequently, by using $(P1)(i)$, $M = \{o\}$,

$(Tr\Lambda)$ $\Lambda(a, b) = \Lambda(o, o) = \theta^{-1}$ for all $a, b \in M$.

Let M_H be a hypergroup with the system of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$. For any mapping $\kappa : M \to H$, $a \mapsto \kappa_a$ consider the following mappings:

(Φ_κ) $\Phi_\kappa(a, \alpha) = \Phi(a, \alpha)$,

(Ψ_κ) $\Psi_\kappa(a, \alpha) = \kappa_a \cdot \Psi(a, \alpha) \cdot \kappa^{-1}_{\Phi(a,\alpha)} = \kappa_a \cdot {}^a\alpha \cdot \kappa^{-1}_{a^\alpha}$,

(Ξ_κ) $\Xi_\kappa(a, b) = \Xi(\Phi(a, \kappa_b), b) = [a^{\kappa_b}, b]$,

(Λ_κ) $\Lambda_\kappa(a, b) = \kappa_a \cdot \Psi(a, \kappa_b) \cdot \Lambda(\Phi(a, \kappa_b), b) \cdot \kappa^{-1}_{\Xi(\Phi(a,\kappa_b),b)} = \kappa_a \cdot {}^a\kappa_b \cdot (a^{\kappa_b}, b) \cdot \kappa^{-1}_{[a^{\kappa_b},b]}$.

It can be shown ([9], Proposition 3) that the system of structural mappings $\Omega_\kappa = (\Phi_\kappa, \Psi_\kappa, \Xi_\kappa, \Lambda_\kappa)$ satisfies the conditions $(P1) - (P4)$ hence forms a hypergroup structure on M over H. We denote this hypergroup by $(M_H)_\kappa$.

Definition 1.6 The map $\kappa \in H^M$ is called a **scalar**, if for an element $\alpha \in H$ we have $\kappa_a = \alpha$ for all elements $a \in M$. We denote such κ by α^M.

The set of all mappings $\kappa : M \to H$ and the set of all scalars are denoted by H^M and $(H^M)_{const}$ respectively.

Definition 1.7 We say that a hypergroup M_H is **similar** to a hypergroup $(M_H)'$ if there exists a scalar $\kappa \in (H^M)_{const}$ such that $(M_H)' = (M_H)_\kappa$.

Chapter 2: unitary hypergroups over the group. This chapter is devoted to an important class of hypergroups, namely *unitary* hypergroups over the group. This class provides a description of all hypergroups M_H up to isomorphism for a given set M and a group H. Moreover, in this chapter an algorithm is discussed to obtain all hypergroups up to isomorphism from a complete representative set of non-isomorphic unitary hypergroups.

Definition 2.1 A hypergroup M_H is called **unitary hypergroup**, if $\Lambda(o,o)^{-1} = \theta = \varepsilon$, where ε is the neutral element of the group H.

Theorem 2.1 ([11], Theorem 2.1) For any hypergroup M_H there exists a unique unitary hypergroup $(M_H)'$, similar to M_H. In this case if $(M_H)' = (M_H)_\kappa$, then $\kappa = \theta^M$.

Let M_H and $M'_{H'}$ be arbitrary hypergroups over the group, $f_0 : H \to H'$ be a homomorphism of groups, $f_1 : M \to M'$ be a map of sets and $f = (f_0, f_1)$.

Definition 2.2 The pair of elements $\kappa \in H^M$ and $\kappa' \in (H')^{M'}$ is called **compatible** with $f = (f_0, f_1)$, if

$$\kappa \circ f_0 = f_1 \circ \kappa' \quad \text{(i.e. } f_0(\kappa_a) = \kappa'_{f_1(a)} \text{ for any } \ a \in M). \tag{4}$$

We denote by $Compatf$ the set of all pairs $(\kappa, \kappa') \in H^M \times (H')^{M'}$, compatible with $f = (f_0, f_1)$. Note that the set H^M (as well as $(H')^{M'}$) together with the operation $*$ (for which $(\kappa * \lambda)_a = \kappa_a \cdot \lambda_a$) forms a group.

Proposition 2.1 ([11], Proposition 4.1) The subset $Compatf$ of the set $H^M \times (H')^{M'}$ determines a subgroup of the direct product of groups H^M and $(H')^{M'}$.

Theorem 2.2 ([11], Theorem 4.2) If f gives a morphism from the hypergroup over the group M_H to the hypergroup over the group $M'_{H'}$ and the pair (κ, κ') is compatible with f, then f gives a morphism from the hypergroup $(M_H)_\kappa$ to the hypergroup $(M'_{H'})_{\kappa'}$, as well.

Theorem 2.3 ([11], Theorem 4.3) Suppose that the pair $f = (f_0, f_1)$ determines a morphism from the hypergroup M_H to the hypergroup $M'_{H'}$. Let $\kappa \in (H^M)_{const}, \kappa' \in ((H')^{M'})_{const}$ be such scalars that f determine a morphism from the hypergroup $(M_H)_\kappa$ to the hypergroup $(M'_{H'})_{\kappa'}$. Then the pair (κ, κ') is compatible with f.

Hence the following theorem holds.

Theorem 2.4 Let $f = (f_0, f_1) : M_H \to M'_{H'}$ be an arbitrary morphism of hypergroups over the group, $\kappa = \alpha^M \in (H^M)_{const}$ and $\kappa' = (\alpha')^{M'} \in ((H')^{M'})_{const}$. Then $f : (M_H)_\kappa \to (M'_{H'})_{\kappa'}$ is a morphism of hypergroups over the group if and only if $f_0(\alpha) = \alpha'$.

Let M_H be a hypergroup over the group H. Denote by $Aut(H, M_H)$ the subgroup of $Aut(H)$ consisting of such automorphisms $f_0 : H \to H$ that there exists a bijection $f_1 : M \to M$ such that (f_0, f_1) is an automorphism of the hypergroup M_H.

There exists a *natural* group action

$$H \times Aut(H, M_H) \to H, \quad (\alpha, f_0) \mapsto f_0(\alpha) \tag{5}$$

of the group $Aut(H, M_H)$ on H. It is a restriction of the natural action of the group

$Aut\,H$ on H. The orbits of this action form a partition of H. We denote this partition by $\mathcal{P}(H, M_H)$.

Recall that for the basic set M and the group H, the set of classes of isomorphic hypergroups is denoted by $\mathcal{H}g(M,H)$. Similarly we denote by $\mathcal{H}g_u(M,H)$ the set of classes of isomorphic unitary hypergroups.

Theorem 2.5 ([11], Theorem 6.1) Suppose that

1) $S_u \subseteq Hg_u(M,H)$ is an arbitrary section of the partition $\mathcal{H}g_u(M,H)$ of the set $Hg_u(M,H)$,

2) $S(M_H) \subseteq H$ is a section of the partition $\mathcal{P}(H, M_H)$ of H for each $M_H \in S_u$. Let $(S(M_H)^M)_{const} \subseteq (H^M)_{const}$ be the set of all scalars α^M with $\alpha \in S(M_H)$.

Then

3) $S = \{(M_H)_\kappa \mid M_H \in S_u,\, \kappa \in (S(M_H)^M)_{const}\} \subseteq Hg(M,H)$ is a section of the partition $\mathcal{H}g(M,H)$ of the set $Hg(M,H)$.

Chapter 3: hypergroup axioms. In this chapter we discuss the question of the independence of the hypergroup axioms, investigate some connections between them, and as a consequence obtain an efficient way to construct a special class of exact products of a given group H and a set M. Additionally a Cayley-type theorem is obtained for right quasigroups with a left neutral element.

Theorem 3.1 ([29], Theorem 1) The axiomatic system of hypergroups over the group, consisted of $(P1),(P2),(P3)$ and the identities $(A1),(A2),(A3),(A4),(A5)$ making up $(P4)$, is independent.

The proof of Theorem 3.1 is done by constructing corresponding models. The following theorems allow the construction of the models.

First let M be a set, H be a group, and $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ be a system of mappings for which the notations $(\Phi)-(\Lambda)$ will be used. Let us define the following mappings.

$$\underline{\Psi} : M \to T_H, \quad \underline{\Psi}(a) = \Psi_a, \quad \text{where} \quad \Psi_a : H \to H, \quad \alpha \mapsto {}^a\alpha, \tag{6}$$

where T_H is the set of all mappings $H \to H$.

$$\underline{\Phi} : H \to T_M, \quad \underline{\Phi}(\alpha) = \Phi_\alpha, \text{ where } \Phi_\alpha : M \to M,\ a \mapsto a^\alpha, \tag{7}$$

where T_M is the set of all mappings $M \to M$.

Theorem 3.2 ([29], Theorem 2) Let M be an arbitrary set, H be a group (with the neutral element ε), $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ be a system of mappings such that Φ is trivial and (M, Ξ) is a group. In that case,

1) if Λ is trivial and $\Lambda(o, o) = \varepsilon$, then for the fulfillment of the axioms $(A1), (A3), (A5)$ and non-fulfillment of the axiom $(P3)$, it is necessary and sufficient that $\underline{\Psi}$ be an antihomomorphism and $\underline{\Psi}(a)$ be a non-surjective homomorphism for any $a \in M$;

2) if Λ is trivial, $\Lambda(o, o) = \varepsilon$, and H is a finite group, then for the fulfillment of the axioms $(P3), (A3), (A5)$ and non-fulfillment of the axiom $(A1)$, it is necessary and sufficient that $\underline{\Psi}$ be an antihomomorphism, $\underline{\Psi}(a)$ be such a bijective function that $\underline{\Psi}(a)(\varepsilon) = \varepsilon$ for any $a \in M$, and $\underline{\Psi}(a)$ be a non-homomorphic function for an element $a \in M$;

3) if Λ is trivial, $\Lambda(o, o) = \varepsilon$, and H is a finite group, then for the fulfillment of the axioms $(P3), (A1), (A5)$ and non-fulfillment of the axiom $(A3)$, it is necessary and sufficient that $\underline{\Psi}$ be a non-antihomomorphism, $\underline{\Psi}(a)$ be a homomorphism for any $a \in M$, and $\underline{\Psi}(o)$ be a bijection;

4) if Ψ is trivial then for the fulfillment of the axioms $(P3), (A1), (A3)$ and non-fulfillment of the axiom $(A5)$, it is necessary and sufficient that elements $\Lambda(a, b)$ be in the center $Z(H)$ of the group H for any $a, b \in M$, and the following identity not be satisfied for some $a, b, c \in M$:

$$(A5)'' \quad (a, b) \cdot ([a, b], c) = (b, c) \cdot (a, [b, c]). \tag{8}$$

Theorem 3.3 ([29], Theorem 3) Let M be an arbitrary set, H be a group and $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ be a system of mappings such that Ψ and Λ are trivial with $\Lambda(o, o) = \varepsilon$. In that case,

1) if (M, Ξ) is a group, then for the fulfillment of the axioms $(P1), (A2), (A4)$ and non-fulfillment of $(P2)$, it is necessary and sufficient that $\underline{\Phi}(\alpha) : (M, \Xi) \to (M, \Xi)$ be a homomorphism for any $\alpha \in H$, $\underline{\Phi}(\varepsilon)$ be the identity mapping of M and $\underline{\Phi}$ not be a homomorphism;

2) if (M, Ξ) is a group, then for the fulfillment of the axioms $(P1), (P2), (A4)$ and non-fulfillment of the axiom $(A2)$, it is necessary and sufficient that $\underline{\Phi}$ be a homomorphism, $\underline{\Phi}(\varepsilon)$ be the identity mapping of M and $\underline{\Phi}(\alpha)$ not be a homomorphism for an element $\alpha \in H$;

3) if Φ is trivial, then for the fulfillment of the axioms $(P2), (A2), (A4)$ and non-fulfillment of the axiom $(P1)$, it is necessary and sufficient that the binary operation Ξ be associative but (M, Ξ) not be a group;

4) if Φ is trivial and $(P1)$ holds, then for the fulfillment of the axioms $(P2), (A2)$ and non-fulfillment of the axiom $(A4)$, it is necessary and sufficient that (M, Ξ) not be a group.

Note that the conditions of Theorem 3.2 immediately follow the fulfillment of the axioms $(P1), (P2), (A2), (A4)$. Similarly the conditions of Theorem 3.3 follow the fulfillment of $(P3), (A1), (A3), (A5)$.

Definition 3.1 Let H be a group, M be a set and

$$\Phi : M \times H \to M,\ \Phi(a, \alpha) := a^\alpha \tag{9}$$

be a (right) group action of H on M, i.e. the hypergroup axiom $(P2)$ holds. The group action Φ is called **effective** if there is no non-trivial element $\alpha \in H$ such that $a^\alpha = a$ for any $a \in M$.

Theorem 3.4 Let H be a group, M be a set, and a system of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ satisfies the axioms $(P1), (P2), (A2), (A4)$. If the group action Φ is effective, then $(A1), (A3), (A5)$ are also satisfied. Moreover, if in addition $o^\alpha = o$ for all $\alpha \in H$ then the condition $(P3)$ is also satisfied.

Definition 3.2 The hypergroup M_H with the system of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ is called **completely reduced** if the group action Φ is **effective**, i.e. $\ker \underline{\Phi} = \{\varepsilon\}$.

Hence Theorem 3.4 gives an efficient way to construct completely reduced hypergroups from a group H and a set M. This leads us to the efficient way of constructing exact products of a group H and a set M *associated with completely reduced hypergroups over the group*.

Theorem 3.5 Let H be an arbitrary group, M be an arbitrary set. To construct all exact products $\overline{G} = H \times M$ associated with completely reduced hypergroups M_H it is sufficient to perform the following steps:

1. Consider all binary operations $\Xi : M \times M \to M$ which induce a right quasigroup structure with a left neutral element on the set M.
2. Consider all effective group actions $\Phi : M \times H \to M$ preserving the left neutral element of the right quasigroup (M, Ξ).
3. For the group monomorphism

$$\underline{\Phi} : H \to S_M, \quad \alpha \mapsto \Phi_\alpha : M \to M, \quad \Phi_\alpha(a) = \Phi(a, \alpha) \tag{10}$$

check the following conditions:

3.1. $R_a \circ \Phi_\alpha \circ R^{-1}_{a^\alpha} \in \underline{\Phi}(H)$ for any $a \in M, \alpha \in H$,

3.2. $R_a \circ R_b \circ R^{-1}_{[a,b]} \in \underline{\Phi}(H)$ for any $a, b \in M$

where $R_a : M \to M$ is the right translation of the right quasigroup (M, Ξ) by the element $a \in M$.

4. Define the mapping $\Psi : M \times H \to H$ as $\Psi(a, \alpha) = \underline{\Phi}^{-1}(R_a \circ \Phi_\alpha \circ R^{-1}_{a^\alpha})$.
5. Define the mapping $\Lambda : M \times M \to H$ as $\Lambda(a, b) = \underline{\Phi}^{-1}(R_a \circ R_b \circ R^{-1}_{[a,b]})$.
6. Define the binary operation

$$\alpha a \cdot \beta b = (\alpha \cdot {}^a\beta \cdot (a^\beta, b))[a^\beta, b] \tag{11}$$

on the set $\overline{G} = H \times M$, which makes $\overline{G}$ a group.

The following result is a corollary from Theorem 3.4. It (with a different formulation) can also be found in [3] (Theorem 1.1 (a)) or in [20] (Theorem 3.4). As a consequence we get a Cayley-type theorem for right quasigroups with a left neutral element.

Theorem 3.6 Let (M, Ξ) be a right quasigroup with a left neutral element o. There exist a group H and mappings Φ, Ψ, Λ such that M_H is a completely reduced hypergroup with the system of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$.

Corollary 3.6.1 (Cayley type theorem) Let Q be a right quasigroup with a left neutral element. Then there exist a permutation group G, a subgroup $H \leq G$ and a (right) transversal $M \subseteq G$ of H such that Q is isomorphic to the groupoid (M, Ξ), where

$$\Xi : M \times M \to M, \quad \Xi(a, b) = c \quad \text{such that} \quad c \in M \cap H(a \cdot b). \tag{12}$$

Chapter 4: normal and completely stable subgroups of a group. In this chapter we discuss another application of the concept of hypergroups over the group. We show the equivalence between the concepts of normal and completely stable subgroups of a group.

Definition 4.1 The subgroup H in the group G is called **completely stable**, if all right transversals M with the right quasigroup operation Ξ are isomorphic to each other as right quasigroups with a left neutral element.

Remark A similar definition of perfectly stable subgroup of a group is given in [21]. According to [21] the subgroup $H \leq G$ is called perfectly stable if all right transversals (M, Ξ) such that $M \cap H = \{\varepsilon\}$ are isomorphic to each other.

Proposition 4.1 Let G be a group, H be a normal subgroup of G. Then H is a completely stable subgroup of G.

Theorem 4.1 ([30], Theorem 2) The subgroup H of the group G is completely stable if and only if H is a normal subgroup.

Remark Let us note that in [21] it is shown that for finite groups G the concepts of perfectly stable and normal subgroup are equivalent. However the question about infinite groups still remains open. Hence for groups where the concepts of perfectly and completely stable subgroups are equivalent our result generalizes Lal's theorem for the infinite case.

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Ամփոփում

Ատենախոսությունում ուսումնասիրվել են խմբի նկապմամբ որոշված հիպերխմբերը, սպացվել են հետևյալ արդյունքները:

1. Խմբի նկապմամբ որոշված հիպերխմբերը նկարագրված են իզոմորֆության ճշտությամբ՝ խմբի նկապմամբ որոշված ունիփար հիպերխմբերի իզոմորֆության դասերի միջոցով: Ավելի ճշգրիփ՝ դիցուք

   1) $S_u \subseteq Hg_u(M,H)$-ը $Hg_u(M,H)$ բազմության $\mathcal{H}g_u(M,H)$ փրոհման կամայական հապույթ է,

   2) $S(M_H) \subseteq H$-ը H-ի $\mathcal{P}(H, M_H)$ փրոհման որևէ հապույթ է կամայական $M_H \in S_u$-ի համար: Դիցուք $(S(M_H)^M)_{const} \subseteq (H^M)_{const}$-ը բոլոր այնպիսի α^M սկալարների բազմությունն է, որ $\alpha \in S(M_H)$:

   Այդ դեպքում

   3) $S = \{(M_H)_\kappa \mid M_H \in S_u, \kappa \in (S(M_H)^M)_{const}\} \subseteq Hg(M,H)$-ը $Hg(M,H)$ բազմության $\mathcal{H}g(M,H)$ փրոհման հապույթ է:

2. Ցույց է փրված խմբի նկապմամբ որոշված հիպերխմբերի $(P1),(P2),(P3)$ և $(P4)$-ը կազմող $(A1)-(A5)$ աքսիոմների անկախությունը:

3. Ապացուցված է թեորեմ, ըսփ որի՝ $(P3),(A1),(A3),(A5)$ աքսիոմները բխում են $(P1),(P2),(A2),(A4)$ աքսիոմներից, Φ խմբային գործողության էֆեկփիվությունից և $\Phi(o,\alpha) = o$ պայմանից՝ կամայական $\alpha \in H$ փարրի համար:

4. Սփացված է մեթոդ, ըսփ որի կարելի է կառուցել նոր խմբեր՝ ելնելով փրված H խմբից և (M,Ξ) ձախ չեզոք փարրով աջ քվազիխմբից: Այդ նոր խմբերն անվանում ենք H խմբի ընդլայնումներ (M,Ξ) ձախ չեզոք փարրով աջ քվազիխմբով:

5. Ապացուցված է Քելիի փիպի թեորեմ ձախ չեզոք փարրով աջ քվազիխմբերի համար՝ օգփվելով խմբի նկապմամբ որոշված հիպերխմբերի գաղափարից: Ավելի ճշգրիփ՝ դիցուք Q-ն ձախ չեզոք փարրով աջ քվազիխումբ է, այդ դեպքում գոյություն ունի փեղադրությունների այնպիսի G խումբ, նրա $H \leq G$ ենթախումբ և H-ի (աջ) փրանսվերսալ $M \subseteq G$, որ Q-ն իզոմորֆ է (M,Ξ) խմբակերպին, որտեղ

$$\Xi : M \times M \to M, \quad \Xi(a,b) = c \quad \text{այնպես, որ} \quad c \in M \cap H(a \cdot b). \qquad (13)$$

6. Ցույց է փրված նորմալ և կապարելապես սպաբիլ ենթախմբերի հասկացությունների համարժեքությունը:

Резюме

Диссертация посвящена гипергруппам над группой. Получены следующие результаты.

1. Дано описание гипергрупп над группой с точностью до изоморфизма с помощью изоморфных классов унитарных гипергрупп над группой. Точнее, допустим

   1) $S_u \subseteq Hg_u(M,H)$ сечение разбиения $\mathcal{H}g_u(M,H)$ множества $Hg_u(M,H)$,

   2) $S(M_H) \subseteq H$ сечение разбиения $\mathcal{P}(H,M_H)$ множества H для произвольного $M_H \in S_u$. Пусть $(S(M_H)^M)_{const} \subseteq (H^M)_{const}$ множество таких скаляров α^M, что $\alpha \in S(M_H)$.

   Тогда

   3) $S = \{(M_H)_\kappa \mid M_H \in S_u,\, \kappa \in (S(M_H)^M)_{const}\} \subseteq Hg(M,H)$ есть сечение разбиения $\mathcal{H}g(M,H)$ множества $Hg(M,H)$.

2. Доказана независимость аксиом гипергрупп над группой, точнее независимость $(P1), (P2), (P3)$ и $(A1) - (A5)$ образующие $(P4)$.

3. Доказана теорема по которой аксиомы $(P3), (A1), (A3), (A5)$ следуют из аксиом $(P1), (P2), (A2), (A4)$, еффективности действия Φ и условия $\Phi(o,\alpha) = o$, для каждого элемента $\alpha \in H$.

4. Получен метод по которому можно построить новые группы исходя из группы H и правой квазигруппы с левым нейтральным элементом (M,Ξ).

5. Доказана теорема типа Келли для правых квазигрупп с левым нейтральным элементом. Точнее, пусть Q правая квазигруппа с левым нейтральным элементом. Тогда существуют такая группа подстановок G, ее субгруппа $H \leq G$ и (правый) трансверсаль $M \subseteq G$, что Q изоморфна группоиду (M,Ξ), где

$$\Xi : M \times M \to M, \quad \Xi(a,b) = c \quad \text{так, что} \quad c \in M \cap H(a \cdot b). \tag{14}$$

6. Доказана эквивалентность понятий нормальных и совершенно стабильных подгрупп.